Three classical problems relating to E_n-coalgebras in topological spaces

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Abstract

The proposed project brings together three conjectures relating to E_n-coalgebras in spaces.

1 Introduction

The work of J. P. May on algebras over the little n-discs operad $\mathbb{D}_n$ [May06] provides us with both a unifying perspective on cohomology operations and concrete models for homotopy commutativity. Recently, Moreno-Fernández and Wierstra [MW19] have developed an Eckmann-Hilton dual theory for the dual notion of coalgebras over $\mathbb{D}_n$. In this theory, suspensions play the role of loop spaces. This approach also sheds some light on homotopy groups of spheres, as we discuss in section 3.

The proposed PhD project is generally themed around E_n-coalgebras. In many contexts-particularly geometry, these appear to be more natural than their algebraic counterpart. For example, topological spaces are naturally a commutative such coalgebra which make singular chains of topological spaces have the structure of E_∞-coalgebras. Unfortunately, the theory of coalgebras over operads is less gentle than that of algebras. For example, there are examples of non-trivial operads with no non-trivial coalgebras at all [GL20]. Another problem is that the cofree-forgetful adjunction goes in the wrong direction for us to be able to apply the Transfer Principle. It thus becomes difficult to define canonical model structures on coalgebras, even when cofree coalgebras exist.

The purpose of our project is to employ E_n-coalgebras to study three classical problems in algebraic topology. The first of these is homotopy operations, the second is the Hochschild-Konstant-Rosenberg theorem and the last one is Mandell's Theorem.

In Section 2, we discuss Moreno-Fernández–Wierstra theory of coalgebras in topological spaces. Section 3 is about applying this theory to homotopy operations. In Section 4, we briefly discuss the Hochschild-Kostant-Rosenberg theorem and extensions thereof. Finally, in section 5, we discuss Mandell's theorem and our proposed dualization of it.

2 Coalgebras over the little n-discs operad

It is difficult to define cooperads in topological spaces, because Top lacks any notion of base field, and thus that of a dual space. We thus must make do with coalgebras over operads. This is defined as follows. First, we define the coendomorphism operad $\mathrm{CoEnd}(X)$ of a topological space X to be the operad whose $\mathbb{S}$-module structure in arity n is the set of continuous maps from X to $X^{\vee n}$, equipped with the obvious $\mathbb{S}_n$-action and composition maps. A $\mathbb{D}_n$-coalgebra structure is then an operad morphism $\theta : \mathbb{D}_n \to \mathrm{CoEnd}(X)$. One can easily show that n-fold suspensions are naturally equipped with such a structure via the pinch map and this naturally induces the Eckmann-Hilton dual of a Browder bracket.

In his master thesis, supervised by Felix Wierstra and Grégory Ginot, FLynn-Connolly established a similar result in the category of simplicial sets. In this setting, defining the coendomorphism operad is made much more complex by the fact that simplicial sets are rarely fibrant. In particular, even if one starts with a fibrant simplicial set X, the wedge product $X^{\vee n}$ is essentially never a Kan complex. Therefore, one has to make use of a fibrant replacement functor. There are two natural choices for this. One can either use the singular chains on the geometric realisation, or Kan's Ex^{∞} functor. It was shown that simplicial suspensions admit an E_n-structure.

3 Applications to primary homotopy operations

We can also use $\mathbb{D}_n$ to study classical homotopy operations from a new perspective. In the cohomological case [May70], it known that we can produce all primary operations this way. Because homotopy operations come in only three flavours [Sto90]; Whitehead brackets, homotopy groups of spheres and actions of the fundamental group on the higher homotopy groups, this produces some results about the homotopy groups of spheres. The setup is as follows. As the n-sphere is an E_n-coalgebra, we have a coalgebra map

$$D_n(k) \times S^n \to (S^n)^{\vee k}$$

This map is equivariant under the action of $\mathbb{S}_k$ and we can therefore take coinvariants to produce a map.

$$B_n(k) \times S^n \to S^n$$

Here, $B_n(k)$ is unordered configuration space. We can then observe that $B_n(k) \times \{*\}$ lies in the kernel of this map and so we can quotient out by it. One has a homotopy equivalence

$$\Sigma^n(B_n(k)) \vee S^n \cong \frac{B_n(k) \times S^n}{B_n(k) \times \{*\}}$$

where $\Sigma^n(-)$ denotes n-fold suspension. Therefore, one has a canonical map

$$\phi_n : \Sigma^n(B_n(k)) \to S^n$$

In principle at least, homotopy operations are given by elements of the homotopy groups of $\Sigma^n(B_n(k))$, but this is unsatisfactory for computational purposes. In practice, many interesting homotopy operations are given by *n-stable equivariant cells of $D_n(k)$*. That is, given an equivariant CW-decomposition of $D_n(k)$, those cells that are attached via a map that becomes equivariantly null-homotopic after n-suspensions. One can show that (for $n > 0$) this is a property preserved by operadic composition.

Example 3.1. For $n = 2$, we have homotopy equivalences $D_n(2) \cong S^{n-1}$ and $B_n(2) \cong \mathbb{R}P^{n-1}$. There is always a degree 2 map $[2] : S^{n-1} \to \mathbb{R}P^{n-1}$. The composition

$$S^{2n-1} \xrightarrow{\Sigma^n([2])} \Sigma^n(\mathbb{R}P^{n-1}) \xrightarrow{\phi_n} S^n$$

represents the Whitehead element of type (n, n) composed with the fold map (the Whitehead element on its own can be produced via a very similar construction), and for $n = 1, 3, 7$ there is a stable splitting

$$\Sigma^n(\mathbb{R}P^{n-1}) = \Sigma^n(\mathbb{R}P^{n-2}) \vee S^{2n-1}.$$

In these cases the compositions

$$S^{2n-1} \xrightarrow{i} \Sigma^n(\mathbb{R}P^{n-1}) \xrightarrow{\phi_n} S^n$$

where i is the inclusion $S^{2n-1} \hookrightarrow \Sigma^n(\mathbb{R}P^{n-1})$, are the classical Hopf maps.

We conjecture that one can obtain all primary homotopy operations in this way.

Conjecture 1. *All fundamental Whitehead classes and elements of the homotopy groups of spheres may be expressed as n-stable equivariant cells of $D_n(k)$ for some $n, k \in \mathbb{N}$.*

This conjecture turns out to be connected to a dual version of May's recognition principle.

Conjecture 2. *Let X be a $(1 + n)$-connected topological space possessing a $\mathbb{D}_n$-coalgebra structure. Then it has the weak homotopy type of the n-fold suspension of some 1-connected pointed space Y.*

4 The Hochschild–Kostant–Rosenberg Theorem

Hochschild (co)homology [Hoc45] is a (co)homology theory for associative algebras over rings. It turns out to have a multitude of uses in geometry, topology and even physics. For example, the second cohomology group controls the deformation theory of associative algebra. The classical HKR theorem acts as a bridge between homological algebra and geometry, and is very closely related to Kontsevich formality.

Theorem 4.1. *Let $\Bbbk$ be a field and let A be a commutative $\Bbbk$-algebra which is essentially of finite type and smooth over $\Bbbk$. Then there is an isomorphism of graded $\Bbbk$-algebras*

$$\Phi : HH_*(A, A) \xrightarrow{\sim} \Omega^*(A/\Bbbk))$$

Dually, there is an isomorphism between the exterior algebra of derivations and the Hochschild cohomology

$$HH^*(A, A) \cong \Lambda^*(\mathrm{Der}_{\Bbbk}(A, A))$$

A higher order version of Hochschild cohomology was introduced by Pirashvili in [Pir00]. In this theory, E_∞-algebras assume the role of commutative algebras. The infinity category of E_∞-algebras is enriched in simplicial sets via

$$\mathrm{Map}_{E_\infty-\mathrm{alg}}(A, B)_n = \mathrm{Hom}(A, B \otimes C^*(\Delta^n)),$$

and has all ∞-colimits. In particular, one can therefore define the tensor product $A \boxtimes X_\bullet \in E_\infty-\mathrm{alg}$ of an E_∞-algebra A and a simplicial set $X_\bullet$. This means that for every E_∞-algebra B there is a natural equivalence

$$\mathrm{Map}_{E_\infty-\mathrm{alg}}(A \boxtimes X_\bullet, B) \cong \mathrm{Map}_{\mathrm{sSet}}(X_\bullet, \mathrm{Map}_{E_\infty-\mathrm{alg}}(A, B))$$

When A is a CDGA, one can choose a model such that the E_∞-algebra $A \boxtimes X_\bullet$ is a CDGA, which admits a very concrete description [Gin17]. One can use this to show the following

Proposition 4.2. *Let A be a CDGA over a field of characteristic 0. Then $A \boxtimes S^1$ is usual Hochschild chain complex $C^*(A, A)$, where S^1 is the usual simplicial model for the unit circle.*

We are now in a position to state the generalized HKR conjecture.

Conjecture 3. *Let X be a formal simplicial set of finite type in each degree. Let A be a CDGA. Suppose that $(\mathrm{Sym}(V), d)$ is a cofibrant, quasi-free resolution of A. Then there is a natural equivalence*

$$A \boxtimes X \xrightarrow{\sim} \mathrm{Sym}(V \otimes H_*(X), d_X)$$

We call $\mathrm{Sym}(V \otimes H_*(X), d_X)$ the higher X-shaped tangent complex of $A \cong (\mathrm{Sym}(V), d)$. Moreover, if $f : X \to Y$ is a formal map, we have a homotopy commutative diagram

$$\begin{tikzcd}
A \boxtimes X \arrow[r, "\sim"] \arrow[d, "id \boxtimes f"'] & (\mathrm{Sym}(V \otimes H_*(X)), d_X) \arrow[d, "\mathrm{Sym}(id \times H(f_*))"] \\
A \boxtimes Y \arrow[r, "\sim"] & (\mathrm{Sym}(V \otimes H_*(Y)), d_Y)
\end{tikzcd}$$

The differential d_X is defined as follows. Let $\Delta^{(n-1)}$ be the iterated coproduct on $H_*(X)$. Then, using Sweedler notation, define

$$\Delta^{(n-1)}(\alpha) = \Sigma \alpha_{(1)} \otimes \cdots \otimes \alpha_{(n)}$$

and, for $v \in V$ let

$$d(v) = \Sigma v_{(1)} \cdots v_{(n)}$$

where $v_{(1)}, \ldots v_{(n)} \in V$. Then d_X is the unique derivation extending the product

$$d_X(v \otimes \alpha) = \Sigma(v_{(1)} \otimes \alpha_{(1)}) \cdots (v_{(n)} \otimes \alpha_{(n)}))$$

The above statements may be dualised.

Remark 4.3. Ginot has shown this in the case where X is an n-sphere.

Remark 4.4. The tensor product

$$\boxtimes : E_\infty-\text{alg} \times \text{sSet} \to E_\infty-\text{alg}$$

ought to factor through E_∞-coalgebras. This should allow us to make the dependence of the HKR theorem on formality very explicit.

Remark 4.5. The homology of X is naturally a cocommutative coalgebra. The tensor product $V \otimes H_*(X)$ is therefore the tensor product of an algebra and coalgebra (recall that the category of algebras is enriched over coalgebras).

Remark 4.6. The infinity category of CDGAs over a commutative ring R containing the rationals is equivalent to the infinity category of E_∞-algebras of chain complexes over R (see, for instance [Lur17, Proposition 7.1.4.11]). Therefore, if we work rationally, we can replace the word CDGA everywhere above with the word E_∞-algebra.

Motivation 4.7. One of the motivation for studying the higher HKR quasi-isomorphism comes from derived algebraic geometry and mathematical physics. Indeed, in this context one can define derived n-Poisson (dg-)schemes (or stacks)[Pan+13] X, which is a data for instance provided by (the observables of) a n-dimensional quantum field theory. The classical Kontsevich theorem is the $n = 1$-case.

In this higher context one aims to deform the sheaf of functions $\mathcal{O}_X$ into an E_1-algebra structure on $\mathcal{O}_X[[\hbar]]$ (in the smooth affine case) or rather to deform its symmetric monoidal category of left modules into an E_{n-1}-monoidal category locally equivalent to modules over an E_n-deformation of $\mathcal{O}_X[[\hbar]]$. The higher Hochschild cochain complexes and their E_{n+1}-structure are the complexes controlling those deformations while their cohomology are precisely the higher analogues of polyvector-fields for higher Poisson structures.

Application 4.8. Let M be a topological space and A be a cdga rational model for M. There is a canonical algebra map

$$C^*(\text{Map}(X, M)) \longrightarrow A \boxtimes X$$

induced by the Chen iterated integrals, see [GTZ10]. This map is an equivalence when M is $\dim(X)$-connected. There is a dual map from chains on the mapping space to higher Hochschild cochains as well.

One of the goals of the project is to give explicit models for the above mapping spaces $\text{Map}(X, M)$ as well as the (partial) $n+1$-dimensional field theories structure carried away by their singular chains when X is of dimension n induced by higher string topology operations (aka brane topology), in particular to give an explicit description of those E_{n+1}-structure in terms of the cotangent complex $Sym(V \otimes H_*(S^n))$.

5 Mandell's Theorem

This latter project is to be studied at the ned of the thesis. Much of algebraic topology for the last century been concerned with the quest for ever more complete algebraic invariants of spaces. The traditional picture - cohomology groups, cup products and Steenrod operations - has recently been superseded by the more general structure of algebras over operads. A key development in this direction occurred in 2006, when Mandell showed that this perspective provides us with a complete invariant of spaces.

First, recall that the integral cochains complex of a space X is equipped with the structure of an E_∞-algebra via the action of the operad EZ whose $\mathbb{S}$-module structure in arity n is the set of natural transformations between the functors $C^*(-, \mathbb{Z})^{\otimes n}$ and $C^*(-, \mathbb{Z})$ and with composition performed in the obvious way. Then Mandell's theorem states

Theorem 5.1. *Two finite type, nilpotent spaces X and Y are weakly equivalent if and only if the integral cochain complexes $C^*(X, \mathbb{Z})$ and $C^*(Y, \mathbb{Z})$ are quasi-isomorphic as E_∞-algebras.*

The final step of our proposed project is to dualise the proof of Mandell's Theorem. The singular chain complex of a space is equipped with the structure of a algebra over the cooperad whose $\mathbb{S}$-module structure in arity n is the set of natural transformations between the functors $C_*(-, \mathbb{Z})$ and $C_*(-, \mathbb{Z})^{\oplus n}$ and with composition performed in the obvious way.

Remark 5.2. To tie this back to the work of Moreno-Fernández and Wierstra, recall that $C_*(-, \mathbb{Z})^{\oplus n}$ is quasi-isomorphic to $C_*((-)^{\vee n}, \mathbb{Z})$.

We thus have the following conjecture.

Conjecture 4. *Two nilpotent CW-complexes of finite type X and Y are weakly equivalent if and only if the chain complexes $C_*(X, \mathbb{Z})$ and $C_*(Y, \mathbb{Z})$ are quasi-isomorphic as E_∞-coalgebras.*

Remark 5.3. Conjecture 4 is not simply the linear dual of Mandell's theorem because the singular (co)chain complex is not finitely generated. In fact, this conjecture is likely to be very difficult to prove.

References

[Hoc45] G. Hochschild. "On the cohomology groups of an associative algebra". In: *Ann. of Math. (2)* 46 (1945), pp. 58–67. ISSN: 0003-486X. DOI: `10.2307/1969145`. URL: `https://doi.org/10.2307/1969145`.

[May70] J. Peter May. "A general algebraic approach to Steenrod operations". In: *The Steenrod Algebra and its Applications (Proc. Conf. to Celebrate N. E. Steenrod's Sixtieth Birthday), Battelle Memorial Inst., Columbus, Ohio* (1970).

[Sto90] Christopher R. Stover. "A van Kampen spectral sequence for higher homotopy groups". In: *Topology* 29.1 (1990), pp. 9–26. ISSN: 0040-9383. DOI: `10.1016/0040-9383(90)90022-C`. URL: `https://doi.org/10.1016/0040-9383(90)90022-C`.

[Pir00] Teimuraz Pirashvili. "Hodge decomposition for higher order Hochschild homology". In: *Ann. Sci. École Norm. Sup. (4)* 33.2 (2000), pp. 151–179. ISSN: 0012-9593. DOI: `10.1016/S0012-9593(00)00107-5`. URL: `https://doi.org/10.1016/S0012-9593(00)00107-5`.

[May06] J Peter May. *The geometry of iterated loop spaces*. Vol. 271. Springer, 2006.

[GTZ10] Grégory Ginot, Thomas Tradler, and Mahmoud Zeinalian. "A Chen model for mapping spaces and the surface product". en. In: *Annales scientifiques de l'École Normale Supérieure* Ser. 4, 43.5 (2010), pp. 811–881. DOI: `10.24033/asens.2134`. URL: `http://www.numdam.org/item/ASENS_2010_4_43_5_811_0/`.

[Pan+13] Tony Pantev et al. "Shifted symplectic structures". In: *Publ. Math. Inst. Hautes Études Sci.* 117 (2013), pp. 271–328. ISSN: 0073-8301. DOI: `10.1007/s10240-013-0054-1`. URL: `https://doi.org/10.1007/s10240-013-0054-1`.

[Gin17] Grégory Ginot. "Hodge filtration and operations in higher Hochschild (co)homology and applications to higher string topology". In: *Algebraic topology*. Vol. 2194. Lecture Notes in Math. Springer, Cham, 2017, pp. 1–104.

[Lur17] Jacob Lurie. *Higher algebra*. unpublished, 2017. URL: `https://www.math.ias.edu/~lurie/papers/HA.pdf`.

[MW19] José M Moreno-Fernández and Felix Wierstra. "Iterated suspensions are coalgebras over the little disks operad". In: *arXiv preprint arXiv:1909.11043* (2019).

[GL20] Brice Le Grignou and Damien Lejay. *Operads without coalgebras*. 2020. arXiv: `1902.02551 [math.AT]`.